# THE SOLUTION OF E.P.R. PARADOX IN QUANTUM MECHANICS

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**Abstract.** We realize a Clifford bare-bone-skeleton of quantum theory showing that, with regard to it, some important mathematical and physical features are missing in standard formulation of quantum mechanics. On this basis, the E.P.R. paradox of quantum mechanics is analyzed, and it is solved and explained.

## 1. Introduction

As we know, quantum theory started in 1927 with extraordinary physical intuitions but it is generally little known that this theory sustained a rather suffered pattern to arrive finally to the identification of the most suitable mathematical framework for its correct theoretical formulation.

There is a book of Jagdish Mehra and Helmut Rechemberg entitled The Historical Development of Quantum Theory [1]. It is an excellent publication on the historical formulation of quantum mechanics. These authors affirm that when Heisenberg "hit upon the non commutativity property of quantum-theoretical quantities, did not know that there existed sophisticated algebraic methods, which in fact had been perfected by Hilbert and his collaborators in Gottingen, and which could be adapted to handle his problems".

In their book Mehra and Rechenberg seem to indicate that also quaternions could have had a role in the mathematical-theoretical elaboration of quantum theory. In fact in the book they consider the noble origin of the quaternion applications in physics. They remember that Hamilton applied quaternions to lunar theory, J.C. Maxwell wrote his Treatise on Electricity and Magnetism in 1873 with regard to quaternions, A.P. Guthrie attempted to persuade his contemporanies to make use of quaternions in physics and, in particular, his friend William Thomson (Lord Kelvin) acknowledged the importance of such numbers in physical studies and the extraordinary beauty of Hamilton's genius to this regard.

P.A.M. Dirac contributed at the highest level to the final formulation of quantum mechanics. In his [2] "The Development of Quantum Theory" and in "History of Twentieth Century Physics", he wrote:

"I saw that non commutation was really the dominant characteristic of Heisenberg's new theory. It was really more important than Heisenberg's idea of building up the theory in terms of quantities closely connected with experimental results. So I was led to concentrate on the idea of non commutation……. . I was dealing with these new variables, the quantum variables, and they seemed to be some very mysterious physical quantities and I invented a new word to describe them. I called them q-numbers and the ordinary variables of mathematics I called c-numbers to distinguish them. The letter q…… stands for quantum and the letter c…… stands for classical. Then I proceed to build up a theory of these q-numbers. Now, I did not know anything about the real nature of q-numbers. Heisenberg's matrices **I thought were just an example of q-numbers, may be q-numbers were really something more general**. All that I knew about q-numbers was that they obeyed an algebra satisfying the ordinary axioms except for the commutative axiom of multiplication …… . I did not bother at all about finding a **precise mathematical nature of q-numbers".**

There is another excellent book published by T.F. Jordan [3] in 1985 entitled "Quantum Mechanics in Simple Matrix Form". Also according to Dirac's position that Heisenberg's matrices **were just an example of q-numbers**, while instead the precise mathematical nature of such q-numbers should be found, we consider Jordan's book of extraordinary importance. In fact, this author uses matrices to express relations between quantum physical quantities. The most important feature is that Jordan arrives to discuss a kind of bare-bones skeleton of quantum mechanics. This is so important since he then uses such bare-bones skeleton of quantum mechanics to perform calculations, and re-obtaining all the standard results of the traditional quantum mechanics that, we know, is formulated by using linear hermitean operators in Hilbert space. Using the same approach Jordan [4] also arrives to explore the basic quantum mysteries in following papers.

In a recent book published by us [5] and entitled Biquaternion Quantum Mechanics, we have followed step by step the indications, the methodology, the teaching, and the results previously given by T.F. Jordan in his book but having this time a precise finality: to realize a bare-bones skeleton of quantum mechanics by using Clifford algebra and thus to conclude that the precise mathematical nature of the q-numbers, as it was suggested from Dirac, should be represented from members of Clifford algebra.

In the present paper we will attempt to give a rigorous exposition of our formulation discussing, in particular, one of the most interesting paradox in quantum mechanics: the so called Einstein-Podolsky-Rosen (E.P.R.) paradox.

Before to conclude this section, we intend to clear the reasons that, in our opinion, give great importance to our effort to express at least a Clifford bare-bones skeleton of a quantum theory. As previously stated, quantum mechanics started in 1927 on the basis of some extraordinary physical intuitions and it arrived to be formulated by a mathematical framework that in time has revealed and confirmed all its power in predicting the results of the theory. Thus, in principle, we should not have good reasons to discuss the utilization of other mathematical instruments and number fields in order to re-derive consolidate results of quantum theory. Consequently, one should not even have in principle good reasons to search for other bare-bones skeletons of quantum mechanics. Instead, we see a good reason in what it follows. As we know, any exposition of conventionally formulated quantum mechanics starts always with the use of classical analogies. **We retain that this is the most questionable point of all the formulation of quantum theory**. As well as we use classical analogies, we attempt to negate the fundamental quantum nature of microphysical reality

that is fixed on three basic and unclassical features: it has integer quanta, it has non commutation and it has indeterminacy. The net distinction that was suggested by Dirac between q- and c- numbers, no more results so definitive when we use classical analogies to introduce quantum theory. In consequence, if for the first time, we would be able to obtain that all the features of quantum reality, that are quantization, non cummutation, indeterminacy are actually due to the outset of the basic axioms of Clifford algebra, we would obtain for the first time an independent and coherent axiomatic representation of quantum reality. This is the basic support of our research that started with the previously considered book [5] and, in our opinion, it receives here a stronger conceptual motivation by the discussion and interpretation of EPR paradox as given in the present paper.

**By this work we outline, in fact, the basic role had from non commutativity of Clifford algebraic members. Non commutation results to be the most fundamental requirement that, without any classical correspondence, we must correctly formulate when we attempt to describe physical systems under the domain of quantum physical reality.**

## 2. Introductory Mathematics

### 2.1 GENERAL STATEMENTS

Let us fix the basic mathematical framework of our paper.

Let us remember that according to P. Lounesto [6], the 3- dimensional Euclidean space $\Re^3$ has a basis consisting of three orthogonal unit vectors $e_1, e_2, e_3$ .

The Clifford algebra $Cl_3$ of $\Re^3$ is the real associative algebra generated by the set $(e_1 , e_2 , e_3)$ satisfying the relations

$$e_1^2 = e_2^2 = e_3^2 = 1$$

and (1)

$$e_1 e_2 = - e_2 e_1 ; \quad e_1 e_3 = - e_3 e_1 ; \quad e_2 e_3 = - e_3 e_2$$

It is 8-dimensional with the following basis

| | |
|---|---|
| $1$ | the scalar |
| $e_1, e_2, e_3$ | vectors |
| $e_1 e_2, e_1 e_3, e_2 e_3$ | bivectors |
| $e_{123} = e_1 e_2 e_3$ | a volume element |

(2)

An arbitrary element in $Cl_3$ is the sum of a scalar, a vector, a bivector, and a volume element, and it must be written in the following manner

$$q = \gamma + \mathbf{a} + \mathbf{b}\, e_{123} + \mu\, e_{123} \tag{3}$$

with $q \in Cl_3$ ; $\gamma, \mu \in \Re$ ,and $a, b \in \Re^3$.

According to the arguments developed in the introduction, as in particular we will discuss also in the following section, we are mainly concerned with the problem of a matrix representation of $Cl_3$. Therefore, let us denote the set of 2 X 2 matrices with complex numbers as entries by Mat(2,C). This set may be regarded also as a real algebra with scalar multiplication taken over the real numbers in $\Re$ also if the matrix entries are in the complex field C.

Let us remember that the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

satisfy the multiplication rules

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

and (5)

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\,\sigma_3$$
$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\,\sigma_2$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\,\sigma_1$$

According still to P. Lounesto [6], the (5) generate the real algebra Mat(2,C).The correspondence

| | | |
|---|---|---|
| $I$ | $\leftrightarrow$ | $1$ |
| $\sigma_1, \sigma_2, \sigma_3$ | $\leftrightarrow$ | $e_1, e_2, e_3$ |
| $\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_1\sigma_3$ | $\leftrightarrow$ | $e_1e_2, e_2e_3, e_1e_3$ |

(6)

establish an isomorphism between real algebras. We have that

$$Cl_3 \cong Mat(2,C) \tag{7}$$

This is the important result that is necessary for the development of our results.

Obviously, it must be outlined that an essential difference between the Clifford algebra $Cl_3$ and Mat(2,C) remains in the fact that in $Cl_3$ we distinguish, by definition, a particular subspace, the vector space $\Re^3$, while no distinguished subspace is signed in the definition of Mat(2,C) (see, still P. Lounesto).

### 2.2 CLIFFORD BASIS MATRIX THEORY

In this paper we support the following thesis: a profound link holds between conceptual and mathematical foundations of quantum physical reality (CMFQM) ) and Clifford algebra.

We attempt to evidence such thesis considering in some detail the following bounds :

CMFQM $\leftrightarrow$ Matrix Algebra $\leftrightarrow$ Clifford Algebra.

Let us examine the first part of such admitted connections.

As we stated in the introduction, T.F. Jordan [3] published an important book in which he formulated quantum mechanics in simple matrix form. The relevant result is that he was able to express the most important conceptual foundations of quantum mechanics, and to re-derive all the standard results of such quantum theory using only the simple mathematical instrument of matrices. Thus, we conclude that this author showed that we may realize quantum mechanics using only this mathematical approach. Obviously, the reader should intend the meaning that we attribute to such conclusion. T.F. Jordan was able to reproduce, by simple matrix approach, a bare-bone skeleton of quantum mechanics without recovering, in particular, the standard mathematical language of this theory that, as we well know, is realized by using linear hermitean operators acting on Hilbert spaces. In this sense, Jordan showed that a profound link exists between CMFQM and matrix algebra. On the other hand, we know that quantum mechanics was repeatedly formulated in matrix form by a very large number of authors, that we cannot quote here for brevity, and, in particular, it was the first mathematical instrument that was employed by Pauli, Jordan, and Heisenberg [1] starting with the advent of quantum mechanics. Thus, it seems that the first link between CMFQM and matrix algebra is well outlined. At this stage, we must not ignore Dirac's position [2] when this authors correctly stated "Heisenberg's matrices I thought were just an example of q-numbers, may be q-numbers were really something more general".
In fact, we cannot ignore that a profound link exists also between matrices and Clifford algebras.
Let us see the manner in which such link is established: the theory of Clifford algebra includes basically the statement that each Clifford algebra is isomorphic to a matrix representation.
R. Ablamowicz [7] and still many authors [8] gave detailed elaboration of this problem. Thus, we may consider as definitive the statement that for any chosen Clifford algebra, there is a matrix representation which is equivalent. This conclusion seems so to close the ring that we formulated: CMFQM $\leftrightarrow$ Matrix Algebra $\leftrightarrow$ Clifford Algebra.
Note that this our thesis does not intend to support that, using Clifford algebra as well as matrix calculus, we obtain a formulation of quantum mechanics that consequently must be considered equivalent to standard and accepted formulation of this theory. The connections CMFQM $\leftrightarrow$ Matrix Algebra $\leftrightarrow$ Clifford Algebra., here identified, only aim to indicate that, following such three connections, we may realize bare-bones skeletons of a quantum theory and that such skeletons, according to the considerations that we developed in the introduction, could result of some utility with regard to the analysis of the unsolved problems that remained for standard quantum mechanical theory.
Let us return now to our basic statement. It says that, for any chosen Clifford algebra, there is a matrix representation that is equivalent, and in this framework, let us attempt to indicate, only from a mathematical viewpoint, what could be the elements to be used in order to characterize a strong bound as CMFQM $\leftrightarrow$ Clifford algebra.
We think that an answer to this question arrives still from the results that Ablamowicz and other authors obtained [7, 8]. They showed that matrices may be derived for each algebra from a choice of idempotents. Idempotents are members of the algebra with the particular property that, when squared, give themselves.
Idempotents represent the mathematical tools that we must use in order to characterize in detail the connection between CMFQM and Clifford algebra.
Let us give an example of Ablamowicz construction for $Cl_2$. It is obviously well known but we retain that it is of some importance for our elaboration to reproduce it here. Of course, the same approach may be followed also in examination of other Clifford algebras that directly result of interest in the present work.
First of all, let us remember that for most Clifford algebras there is at least one primitive idempotent, so that its square gives itself. For $Cl_2$, we have two basic members (basic elements) $e_i$ $(i = 1,2)$ $(e_1 e_2 = - e_2 e_1)$, and one such idempotent involves only one basic element, i.e.,

$$\psi_1 = \frac{1}{2}\,(1 + e_1)\ , \qquad \psi_1\psi_1 = \psi_1 \qquad (8)$$

If the idempotent is multiplied by the other basic element, $e_2$, other functions can be generated, as it follows:

$$\psi_2 = e_2\psi_1 = (\frac{1}{2} - \frac{1}{2}\,e_1)\,e_2\ ; \qquad \psi_3 = \psi_1\,e_2 = (\frac{1}{2} + \frac{1}{2}\,e_1)\,e_2; \qquad \psi_4 = e_2\psi_1 e_2 = \frac{1}{2} - \frac{1}{2}\,e_1. \quad (9)$$

In addition, we have also that

$$\psi_1\,e_1\psi_1 = 0$$

The important thing that we must retain for the following arguments is that the four functions $\psi_i$ (i=1, 2, 3, 4) provide means to represent any member of the space. A general member q is given in terms of the basis members of the algebra in the following manner

$$q = a_0 + a_1\,e_1 + a_2\,e_2 + a_3\,e_1e_2 \qquad (10)$$

and q may be represented by the series of terms of the idempotents

$$q = a_{11}\,\psi_1 + a_{21}\psi_2 + a_{12}\psi_3 + a_{22}\psi_4 \qquad (11)$$

with

$$a_{11}+a_{22} = 2\,a_0 \qquad a_{12} + a_{21} = 2\,a_2 \qquad (12)$$
$$a_{11} - a_{22} = 2\,a_1 \qquad a_{12} - a_{21} = 2\,a_3$$

On the other hand, calculating $\psi_1 q\psi_1$, $\psi_1 q\psi_2$, $\psi_3 q\psi_1$, $\psi_3 q\psi_2$, one finds that a matrix A may be defined

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{13}$$

with

$$a_{11} = a_0 + a_1, \qquad a_{22} = a_0 - a_1, \qquad a_{12} = a_2 + a_3, \qquad a_{21} = a_2 - a_3$$

and

$$(1\ e_2)\ A\ \psi_1 \begin{pmatrix} 1 \\ e_2 \end{pmatrix} = q = a_{11}\psi_1 + a_{21}\psi_2 + a_{12}\psi_3 + a_{22}\psi_4 \tag{14}$$

Thus, we may conclude that the (14) generates the general Clifford number q. When equating q with 1, $e_1$, $e_2$, $e_1e_2$, respectively, we obtain the final matrix expressions multiplied by the idempotent:

$$q = 1, \quad A\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi_1; \qquad q = e_2, \quad A\psi_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi_1$$

$$q = e_1, \quad A\psi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi_1; \qquad q = e_1e_2, \quad A\psi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi_1 \tag{15}$$

These are the usual basis matrices for $Cl_2$.

We have given indication on the manner in which correspondence is realized between basis matrices and Clifford algebra itself: in order to obtain matrix representation of basic elements of the algebra, we must specify an idempotent and the associated vector of basis functions. We have done this for $Cl_2$, and it is easily extended to other algebras once the idempotent, and the vector of basis functions have been identified.

The previous elaboration of Ablamowicz and other authors [7, 8] represents the central core of our formulation from a mathematical as well as conceptual viewpoint.

Let us outline, as example, that a set of basis matrices for $Cl_3$ may be obtained following the same previous procedure. In this case one obtains that

$$e_0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \qquad e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16}$$

With a proper choice of the idempotent, we may also arrive to express new sets of basis matrices at n = 4 , 8 , .... .

They are expressed in the following manner

$$E_{0i} = I^1 \otimes e_i; \qquad E_{i0} = e_i \otimes I^2 \tag{17}$$

The notation $\otimes$ denotes direct product of matrices, and $I^i$ is the ith 2x2 unit matrix. Thus, in the case of n= 4 we have two distinct sets of basis matrices, $E_{0i}$ and $E_{i0}$, with

$$E_{0i}^2 = 1; \qquad E_{i0}^2 = 1; \qquad i = 1, 2, 3; \tag{18}$$

$$E_{0i} E_{0j} = i\, E_{0k}; \qquad E_{i0} E_{j0} = i\, E_{k0}; \quad j = 1, 2, 3; \qquad i \neq j$$

and

$$E_{i0} E_{0j} = E_{0j} E_{i0} \tag{19}$$

with (i, j, k) cyclic permutation of (1, 2, 3).

Let us examine now the following result

$$(I^1 \otimes e_i)(e_j \otimes I^2) = E_{0i} E_{j0} = E_{ji} \tag{20}$$

We have that $E_{0i} E_{j0} = E_{ji}$ with i = 1, 2, 3 and j =1, 2, 3, with $E_{ji}^2 = 1$, $E_{ij} E_{km} \neq E_{km} E_{ij}$, and $E_{ij} E_{km} = E_{pq}$ where p results from the cyclic permutation (i, k, p) of (1, 2, 3) and q results from the cyclic permutation (j, m, q) of (1, 2, 3).

In the case n = 4 we have two distinct basic matrices $E_{0i}$, $E_{i0}$ and, in addition, basic sets of unities ($E_{ij}$, $E_{ip}$, $E_{0m}$) with ( j, p, m) basic permutation of (1, 2, 3). Similarly, we may realize other basic sets of basis matrices using ($E_{ji}$, $E_{pi}$, $E_{m0}$). Note the basic role explained from cyclic permutations of (1, 2, 3) and their strong connection with non commutativity of the chosen basic elements.

In the case of matrix representation at order n= 8, we have the possibility to introduce three sets of basic unities. We will have $E_{00i}$, $E_{0i0}$, and $E_{i00}$, i = 1, 2, 3 and

$$E_{00i} = I^1 \otimes I^1 \otimes e_i; \qquad E_{0i0} = I^2 \otimes e_i \otimes I^2; \qquad E_{i00} = e_i \otimes I^3 \otimes I^3$$

and

$$(I^1 \otimes I^1 \otimes e_i) \cdot (I^2 \otimes e_i \otimes I^2) \cdot (e_i \otimes I^3 \otimes I^3) = e_i \otimes e_i \otimes e_i = E_{00i} E_{0i0} E_{i00} = E_{iii} \tag{21}$$

Still we will have that

$$E_{00i} E_{0i0} = E_{0i0} E_{i00}; \qquad E_{00i} E_{i00} = E_{i00} E_{00i}; \qquad E_{0i0} E_{i00} = E_{i00} E_{0i0} \tag{22}$$

In the case n = 8 we have three distinct basic unities, and, in addition, we have other sets of basis matrices, as example ($E_{ijk}$, $E_{iji}$, $E_{ijj}$). Other cases are obviously possible.

Generally speaking, fixed the order n of the matrix representation of the set of basis matrices, we will have that

$$\Gamma_1 = \Lambda_n$$

$$\Gamma_{2m} = \Lambda_{n-m} \otimes e_2^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n \tag{23}$$

$\Gamma_{2m+1} = \Lambda_{n\text{-}m} \otimes e_3^{(n-m+1)} \otimes I^{(n-m+2)} \otimes \ldots\ldots\ldots \otimes I^n$

$\Gamma_{2n} = e_2 \otimes I^{(2)} \otimes \ldots\ldots\ldots \otimes I^n$

with

$\Lambda_n = e_1^{(1)} \otimes e_1^{(2)} \otimes \ldots\ldots \otimes e_1^{(n)} = (e_1 \otimes I^{(1)} \otimes \ldots\ldots \otimes I^n) \cdot (\ldots\ldots\ldots) \cdot (I^{(1)} \otimes I^{(2)} \ldots \otimes I^{(n)} \otimes e_1)$;

m = 1, ....., n - 1

according to the n-possible dispositions of $e_1$ and $I^1$, $I^2$, ..., $I^n$ in the distinct direct products.

Basis matrices are determined by the number of dispositions possible for $e_i$ and $I^{(n)}$.

In this manner we have established that matrix representations of $Cl_3$ exist at different orders n = 2, 4, 8, ..... .

We may now give the explicit expressions of $E_{0\,i}$, $E_{i\,0}$, and $E_{i\,j}$.

$$E_{01} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad E_{02} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad E_{03} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad E_{20} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}; \quad E_{30} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$E_{11} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad E_{22} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \quad E_{33} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (24)$$

$$E_{12} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \quad E_{21} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix};$$

$$E_{31} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \quad E_{23} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}; \quad E_{32} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

## 2.3 A BARE-BONE SKELETON OF QUANTUM MECHANICS

The first announcement that, using matrix representation of $Cl_3$, we are considering a ground that has profound physical implications at the level of quantum physical reality, arrives when we give a physical identification to the matrix representations that are given in the (16), and, soon after, in the (24) .

Let us explain this point.

As we know, in the traditional formulation of quantum mechanics, we elaborate the theory starting with some analogies that we attempt to establish with classical physics; however, when exposing the standard formulation of the theory, we arrive to a such point of its exposition that, soon after, it becomes impossible for us to continue to admit analogies. At this time, we are forced to admit an existing physical quantity that we consider not having classic analogy: this is the time in which we are forced for the first time to admit the spin for a given quantum mechanical system.

In usual quantum mechanics physical quantities are represented by linear hermitean operators. Spin angular momentum and magnetic moment of a spin ½ particle are introduced. The projections of the spin in three perpendicular reference directions are represented by the operators $S_i = ½\hbar\ e_i$ (i=1,2,3) where, as identified by Pauli, the $e_i$ are just the matrices that we give in the (16). The projections of the magnetic moments in the three perpendicular reference directions are given instead by the operators $\mu_i = \mu\ e_i$ where $\mu$ is a real number. This is the case of only one spin ½ particle. Often, in quantum mechanics, we examine also systems of two spin ½ particles. In this case, the spins of two particles can be measured together, as we know, since they are for different particles. As consequence, the operators, representing the spin of such two particles respectively, must commute. In this case the $e_i$ in the (16)no more may be used to represent the spins of the two particles, and, still by standard arguments, one arrives to consider the operators as they are given this time in the (24), where the first set is attributed to one particle and the other set is attributed to the other particle. Thus, we may conclude that the first and the most important quantum physical quantity, formulated in quantum mechanics without any analogy with classical physics, is the ½ spin of a particle, and it has in quantum mechanics an

operator representation that has a direct correspondence with the matrix representations that we, only mathematically speaking, find also when we explicit matrix representations of $Cl_3$ .
We may say that the profound connection admitted in the previous section between CMFQM and Clifford algebra starts to be evidenced. We must account also for the basic importance, nature, and role of the physical quantity on which we are leaning our CMFQM ↔ Clifford algebra correspondence: it is the spin that, with only few other physical quantities, remains to be the first and the most important non classical observable in the whole body of all the traditional quantum mechanics. Thus, also if our connection CMFQM ↔ Clifford algebra should remain confined to the spin, our possible bare-bone skeleton, realized only on this basis, just should cover a great interest. But, as discussed in detail in [5], one may see that this is not the case, and that we may be able to consider instead the central and universal role of the identified matrix representations of Clifford algebra in all the body of quantum physical processes. At the moment, however, let us remain confined to the analysis of spin, and consider the manner to realize our definitive Clifford-quantum mechanical bare-bone skeleton.
N. Bohr was one of the founder fathers of quantum mechanics. Radically he [9] remarked that basic to quantum theory is the necessity to consider the observed quantum system and the fundamental role of the classical instruments during the process of actual attribution of a numerical value to a quantum observable during a measurement.
We know that the usual quantum mechanics, mathematically based on linear hermitean operators acting on Hilbert spaces, makes, from its starting, a distinction that results of basic importance in order to correctly understand the theory and the correct predictions that it gives. Just using linear hermitean operators connected to physical quantities (observables), quantum mechanics makes a clear distinction between physical quantities and the values that such quantities may assume in the course of a measurement. This is one of the physical and mathematical foundations of quantum mechanics. Consequently, it is evident that, if our attempt is to realize a bare-bone skeleton of quantum mechanics, the first pillar that we must construct, resides just in the crucial respect of this first basic foundation of the traditional theory.
Thus we may return to consider $Cl_3$.
Let us see how it is possible to arrive to attribute a numerical value to one spin component of ½ spin particle, using this time not the standard operators in Hilbert basis but matrix representations of $Cl_3$.
Let us return to our $Cl_3$ mathematical formulation and let us explore still some other important features of the idempotents.
In order to simplify our elaboration we remain to consider matrix representation of $Cl_3$ at order n=2.
Consider that, with regard ,as example, to the basic element $e_3$ of $Cl_3$, we may identify an idempotent $\psi$, and we write that

$$e_3\,\psi = \psi \qquad \text{and} \qquad \psi\, e_3 = \psi \tag{25}$$

Still, we may identify an idempotent $\varphi$ so that

$$e_3\,\varphi = -\varphi \qquad \text{and} \qquad \varphi\, e_3 = -\varphi \tag{26}$$

We may also generalize such definitions.
In fact, we may have that

$$e_3\,\psi_1 = \psi_1 \qquad \text{and} \qquad \psi_2\, e_3 = \psi_2 \tag{27}$$

and

$$e_3\,\varphi_1 = -\varphi_1 \qquad \text{and} \qquad \varphi_2\, e_3 = -\varphi_2 \tag{28}$$

Calculations show that in the case of the (25), we have that

$$\psi = \frac{1+e_3}{2} \tag{29}$$

In the case of the (26), we have

$$\varphi = \frac{1-e_3}{2} \tag{30}$$

In the case of the (27), we have

$$\psi_1 = \frac{1+e_3}{2} + \frac{e_1+ie_2}{2} \qquad \text{and} \qquad \psi_2 = \frac{1+e_3}{2} + \frac{e_1-ie_2}{2} \tag{31}$$

In the case of the (28), we have

$$\varphi_1 = \frac{1-e_3}{2} + \frac{e_1-ie_2}{2} \qquad \text{and} \qquad \varphi_2 = \frac{1-e_3}{2} + \frac{e_1+ie_2}{2} \tag{32}$$

Consider again the (25) that now we rewrite as it follows

$$(e_3 - 1)\,\psi = 0 \qquad \text{and} \qquad \psi(e_3 - 1) = 0 \tag{33}$$

As rigorously admitted from our elaboration of the previous pages, let us calculate now the (33) with regard to $e_i$, i = 1,2 . We obtain that

$$e_i\,(e_3 - 1)\,\psi = 0 \qquad \text{and} \qquad \psi\,(e_3 - 1)\,e_i = 0 \tag{34}$$

Thus, we may conclude, with regard to $\psi$ that, as example, we have that

$$(i\,e_1 - e_2)\,\psi = 0 \qquad \text{and} \qquad \psi\,(i\,e_1 + e_2) = 0 \tag{35}$$

Considering the (26), we may rewrite that

$$(e_3 + 1)\,\varphi = 0 \qquad \text{and} \qquad \varphi\,(e_3 + 1) = 0 \tag{36}$$

We may calculate the (36) with regard to $e_i$, $i = 1,2$, obtaining

$$e_i\,(e_3 + 1)\,\varphi = 0 \qquad \text{and} \qquad \varphi\,(e_3 + 1)\,e_i = 0 \tag{37}$$

Thus we may conclude that with regard to $\varphi$ we have that

$$(i\,e_1 + e_2)\,\varphi = 0 \qquad \text{and} \qquad \varphi\,(i\,e_1 - e_2) = 0 \tag{38}$$

Note that until here we have used only primitive idempotents and generated functions of $Cl_3$ as previously established in our elaboration of the previous pages. All the results, thus, must be considered to be rigorously valid.

Let us now take the meaning of the operations that we have performed until here. It is evident that such operations by idempotents must be seen to be in relation with the corresponding basic foundation of quantum mechanics that we outlined when we considered the profound distinction made from traditional quantum mechanics between a quantum physical quantity and the value that such quantity may assume in the course of a physical measurement. As well as in quantum mechanics we attribute a numerical value to a physical quantity by using a proper eigenvalue equation connected to the linear hermitean operator representing the physical quantity, in correspondence of this, also the Clifford, $Cl_3$, idempotent expressions, given respectively in the (25) and the (26) as well as in the (27) and the (28), with rigorously connected the (35) and the (38), must be intended as if we attribute a numerical value to a basic element, say +1 to $e_3$, of $Cl_3$ in the case of the (25), or –1 to $e_3$ in the case of the (26).

In conclusion, we have actually evidenced the possibility to realize a bore-bone skeleton of quantum mechanics by $Cl_3$. In fact, we have shown that we are in the condition to fix correspondence between quantum physical quantities and basis elements of $Cl_3$; we have also shown the possibility, as in traditional quantum mechanics, to make a net distinction between a quantity and its values, considering that the acquisition of numerical values from the physical quantity, happens only when we consider in $Cl_3$ the idempotents expressions as they have been formulated by us starting with the (25).

The same conclusions we reach also if we decide to attribute definite numerical values (+1 or -1) to the other two basis elements, $e_1$ or $e_2$ and following the corresponding operations previously indicated in the case of $e_3$. In brief, we have realized a Clifford-scheme that, under some features, moves in accord with standard theory.

Note, however, that we have also obtained results that exhibit profound differences respect to standard quantum mechanics. Let us examine again our formulation from the (25) to the (38). For brevity, let us consider only the (25). We may examine it from a mathematical and from a physical viewpoints. In Clifford algebra we consider it as the action of $e_3$ on the idempotent $\psi$. Physically speaking, we are admitting that a quantum physical quantity corresponds to $e_3$ and that, measuring it, we are obtaining one of the two possible values of $e_3$, say + 1. Now the central point is here at a mathematical level. The use of the (25) correctly induces a Clifford algebra that goes from the (25), to the (29) or to the (31), to the (33) and to the (34) with connected the (35). We may say that, starting with the (25), we reduce all the Clifford algebra to the members now indicated. In other terms, we must intend that now all the Clifford algebra is here expressed with regard to the idempotent $\psi$ and to the action of $e_3$ and $e_i$ ($i = 1, 2$) on $\psi$ as given by the (25) until the (35). Introducing the (25), we restrict all the Clifford algebra to only the Clifford members that correctly are given from the (25) to the (35), and every time we consider Clifford members, we must remember that they are expressed only with regard to the idempotent $\psi$ under the action of $e_3$ and $e_i$ ($i = 1, 2$). In the next developments of this paper, we will not mention any time this feature of calculation; however we will express our algebraic notations implicitly intending, any time, that obviously they must be considered with regard to the idempotent employed. In these conditions, the basic arising difference between traditional quantum mechanics and our Clifford bare-bone skeleton is that, as example, the use of the (25) for $e_3$, leads to the to the (35)that instead escapes to the traditional quantum mechanics. Of course, the (35) are rigorously admitted from our Clifford algebra, while instead, traditional quantum mechanics cannot use such operator expressions since it is bounded to the mathematical and physical prescriptions arising from its assiomatic formulation. On the contrary, we consider instead the (35) to explain an algebraic but also an essential physical role in the domain of analysis of quantum physical reality.

In the traditional quantum mechanics a rule of silence holds: given two non commuting observables A and B, a measurement of A, as example, totally prohibits any kind of pronouncement about B. Instead, the conceptual and dynamical effect of non commutativity between A and B still remains in the considered quantum mechanical system also during the measurement of A or of B, and it must be expressed by a mathematical and physical relation. This is correctly seen, instead, in our Clifford bare-bone skeleton of quantum theory where, in fact, it is shown that, during, as example, the numerical attribution to $e_3$; $e_1$ and $e_2$ still reflect their non commuting behaviour respect to $e_3$, as detailed expressed by the (35), and, in more explicit forms, will be expressed by the (48) and the (50). Assuming $e_3$ a precise numerical value, certainly it will result impossible for $e_1$ and $e_2$ to assume simultaneously definite numerical values, however the non commutation between $e_3$ and $e_1$ and $e_2$ still will be required to be clearly characterized at physical level in order to correctly describe the dynamics and the conceptual foundations of the quantum system under consideration.

Let us examine now some consequences of these results with regard to quantum mechanics.

It is evident that, following our scheme, we have now also the possibility to attribute mean values $< e_i >$ ($i = 1, 2, 3$) corresponding to the physical quantities represented by the basic elements $e_i$.

Said p(1) the probability for $e_3$ to assume the value +1 in the course of a measurement, and indicated by p(-1) the probability that it assumes the value –1, as a general rule, we will have that

$$< e_3 > = p(+1)\,(+1) + p(-1)\,(-1) \tag{39}$$

with

$$p(+1) + p(-1) = 1 \tag{40}$$

Generally speaking, we will have that

$$< e_i > = p(+1)\,(1) + p(-1)\,(-1) \tag{41}$$

and i = 1, 2, 3.

Given three constant real or complex numbers $\alpha$, $\beta$, $\gamma$, always we will consider the numerical values $\alpha<e_1>$, $\beta<e_2>$, and $\gamma<e_3>$ and the addition

$$<q> = \alpha< e_1 > + \beta <e_2> + \gamma< e_3> \tag{42}$$

and we will call the mean value $<q>$ connected to the Clifford member $q \in Cl_3$, q being

$$q = \alpha\, e_1 + \beta\, e_2 + \gamma\, e_3 \tag{43}$$

Jordan [3] developed such question for matrices. Consider $q \in Cl_3$

$$q = y_1\, e_1 + y_2\, e_2 + y_3\, e_3 \tag{44}$$

having

$$( y_1\, e_1 + y_2\, e_2 + y_3\, e_3 )\,( y_1\, e_1 + y_2\, e_2 + y_3\, e_3 ) = q^2 = h^2 = y_1^2 + y_2^2 + y_3^2.$$

The mean value $<q>$ must be thus between - h and + h. We have that

$$- h \le < y_1\, e_1 + y_2\, e_2 + y_3\, e_3 > \le + h \tag{45}$$

and thus

$$- h \le y_1 < e_1> + y_2 < e_2> + y_3 < e_3> \le + h \tag{46}$$

The (46) holds for any given set of numbers $y_1, y_2, y_3$, and thus also when we have that

$$y_1 = < e_1> \quad ; \quad y_2 = < e_2> \quad ; \quad y_3 = < e_3> \tag{47}$$

The (48) gives that $h \le 1$, and finally we have that

$$< e_1 >^2 + < e_2 >^2 + < e_3>^2 \le 1 \tag{48}$$

The same relation was found from Jordan [3] for matrices.

Let us admit that we decide to attribute the value +1 to the mean value $< e_3 >$. Assuming the (48), we obtain that

$$< e_3 >^2 = 1 \quad \text{and} \quad < e_2 >^2 + < e_3>^2 = 0 \tag{49}$$

Solving the (49), we have that

$$< e_3 > = 1 \quad \text{and} \quad < e_1> = \pm\, i < e_2> \tag{50}$$

Let us see the very important result that we have obtained in $Cl_3$. It has importance also in our Clifford-bare-bone skeleton of quantum theory: The (48) states that we cannot attribute simultaneous definite numerical values, +1 or - 1, to $e_i$, i = 1, 2 and 3 but only to one of these basic elements for each time. In this manner, we have found also intrinsic indeterminism for the basic elements of $Cl_3$ Clifford algebra. Knowing the foundations of quantum mechanics, mathematically formulated by operators acting on Hilbert spaces, at this point, we may conclude that we have realized actually and rigorously the profound link that we suggested as existing between CMFQM and Clifford algebra. We have actually realized a $Cl_3$ bare-bone skeleton of quantum mechanics also if it is till confined to only one quantum physical quantity. It is, on the other hand, the most important quantum physical quantity under the viewpoint to consider physical quantities that are missing of a classical analogy.

We may also complete our scheme giving a direct and rigorous definition of probabilities that we introduced in the (39) and the (40).

Let us admit that, given an ½ spin system, we intend to measure the third component of the spin to which $e_3$ is connected. It is well known that the traditional quantum mechanics gives definition of the state of the system, and it is characterized by the well known density matrix operator, $\rho$. In our accepted bare-bone skeleton of quantum mechanics, we may still represent $\rho$ by a member of $Cl_3$ that, in fact, will result to be

$$\rho = a + b\, e_1 + c\, e_2 + d\, e_3 \tag{51}$$

where

$$p(+1) + p(-1\,) = 2\, a$$

and (52)

$$p(+1) - p(-1) = 2\, d$$

Let us observe that in the traditional quantum mechanics we have that $\rho^2 = \rho$ for the operator density matrix to represent a quantum system in a pure state; correspondently, in our $Cl_3$ elaboration [10] we have that, in the (51), it is $\rho^2 = \rho$ if and only if a= ½ , as fixed in the (52), and

$$a^2 - b^2 - c^2 - d^2 = 0, \quad \text{that is to say: } \mathrm{Norm}(\rho) = 0.$$

Once again, we find an important connection between traditional quantum mechanics and our Clifford quantum bare-bone skeleton.

Let us analyze, finally, another important feature of our formulation.

Let us re-write the (51) in the following manner

$$\rho = k\left(\frac{1+e_3}{2}\right) + h\left(\frac{1-e_3}{2}\right) + m\left(\frac{e_1+e_2 i}{2}\right) + n\left(\frac{e_1-e_2 i}{2}\right) \tag{53}$$

with

$$k + h = 2\, a \quad ; \qquad m + n = 2\, b;$$
$$k - h = 2d \quad ; \qquad m - n = -\, 2\, i\, c \tag{54}$$

where the idempotents ψ and φ, given respectively in (29) and (30) with respect to $e_3$, now are expressed. Note that, when $e_3$ is considered to assume the values or +1 or –1, according to the (27) or to the (28), also the (35) and (38) must hold, that is to say that, with regard to ψ or φ, we have

$$e_1 + e_2\, i = 0$$

and (55)

$$e_1 - e_2\, i = 0$$

that are also in perfect accord with the (48) and the (50). We see that both the (55) are satisfied for $e_1\, e_2 = i$ with regard to ψ and $e_1\, e_2 = -\, i$ with regard to φ.

To deduce $e_1\, e_2 = i$, let us consider that we are operating with regard to the idempotent ψ. Thus we have that:

$$i\,\psi = i\left(\frac{1+e_3}{2}\right) = e_1\, e_2\, e_3\left(\frac{1+e_3}{2}\right) = e_1\, e_2\left(\frac{1+e_3}{2}\right) \qquad (56)$$

for $e_3$ corresponding to + 1. Similar results may be observed for $e_3$ corresponding to - 1. We have that

$$i\,\varphi = i\left(\frac{1-e_3}{2}\right) = e_1\, e_2\, e_3\left(\frac{1-e_3}{2}\right) = e_1\, e_2\left(\frac{e_3-1}{2}\right) = -\, e_1\, e_2\,\varphi \qquad (56)$$

Inserting the (55) and the (56) in the (53), we see that such $Cl_3$ member ρ is now transformed so that to it we may correspond no more the density matrix operator of a pure state, but that one corresponding instead to a mixture. In other terms, our $Cl_3$ quantum bare-bone skeleton of quantum mechanics describes the well known collapse of the wave function. This is of great importance since the collapse of wave function remained and still remains unsolved in the traditional framework of quantum mechanics.

## 2. The EPR Paradox in Quantum Mechanics

The origin of the topic is well known. It is due to a celebrated paper of Einstein, Podolsky and Rosen (EPR) in 1935 [11]. These authors considered what Einstein called the "spooky action at-distance" that seemed to constitute an essential part of quantum mechanics, and they concluded that this theory must be incomplete if not wrong. Einstein proposed the presence of variables in quantum physical reality whose concrete identification would induce the physics to overcome the intrinsic indetermination exhibited from the quantum mechanical formulation of physical reality. The word "hidden variables" was coined by J. Von Neumann in his book [12] just to characterize Einstein's position. John Bell coined the more neutral term "beables" to indicate those properties that a particle actually possesses independently of our measurements.

Let us define such "**elements of physical reality**" using the same criterion that was introduced by the authors of the formulated paradox: **if, without in any way disturbing a system, we can predict with certainty……. the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity** [11].

Let us now apply this criterion to a composite quantum system consisting of two distant particles. The simplest example of such a situation is that one of two spin-½ particles, far apart from each other, in a singlet state.

Let us comment some well known correlation experiment. As example, electron pairs are emitted by a radioactive substance with a total spin zero. We have that if right hand electron is spin-up, the other electron of each pair is guaranteed to have spin-down provided that the two employed measuring apparata are arranged at the same orientation. Following the EPR argument, we may now fix left hand filter (measuring apparatus) at 45 degrees, and the right hand at zero degrees. This operation seems possible since, according also to our classic logic, we are considering two distinct electrons whose spin values are measured and thus identified in two distinct regions of the space-time.

Let us admit that we identify that the left hand electron passes through its filter and thus has spin-up at orientation 45° degrees. We are guaranteed that the other electron of the pair has spin-down at the same orientation of 45° degrees. We measure simultaneously the right hand electron to determine if it is spin-up at zero degrees. Since no information can travel faster than the speed of light, the left hand measurement will not disturb the right hand measurement. If we find spin-up at zero degrees on the right-hand electron, we will be guaranteed that the other electron of the pair has spin-down at the same orientation of zero degrees. In this manner, it seems that these two simultaneous measurements will enable us to overcome Heisenberg's indetermination principle: in fact, in this case we have simultaneous and definite values of the spin of the right-hand electron at zero degrees and 45 degrees and we have also definite values of the spin of the left-hand electron at zero degrees and 45 degrees.

This was the EPR argument here exposed in a very elementary but significant manner. This was the manner to suggest that quantum mechanics is incomplete if not wrong. We may observe in particular that the EPR argument was developed with the evident intent to show that commutation relations and thus Heisenberg's principle could be overcome.

Now let us explicit the EPR paradox using only standard quantum mechanics. We could follow other general lines to expose here the paradox, obtaining obviously equivalent results, but we prefer to develop the argument as it was previously discussed by A. Peres in 1992 [13] since, in our opinion, it responds to a particular demand of depth and clearness in the elaboration.

According to Peres [13] and thus using standard notations of quantum mechanics, considering the previous system of two spin-½ particles, we have that

$$(E_{01} + E_{10})\,\psi = 0$$

and (57)

$$(E_{02} + E_{20})\,\psi = 0$$

We consider here that $E_{0i}$ (i = 1, 2, 3) are operators regarding (x, y, z) spin components of the first particle while $E_{i0}$ (i = 1, 2, 3) regard (x, y, z) spin components of the second particle.
The first equation asserts that measurements of $E_{01}$ and $E_{10}$, if performed, shall yield opposite values, $c_{01}$ and $c_{10}$ respectively.
According to the previous criterion of Einstein, Podolsky and Rosen, each one of such considered quantum observables will correspond to an "element of reality". In fact its value can be ascertained, without perturbing in any way the particle to which the observable pertains, by means of a measurement performed on the other particle.
The same interpretation may be given with regard to the second equation given in (57). Also in this case we may arrive to identify the numerical values of $E_{02}$ and $E_{20}$ with the aid of only one measurement. Thus the same conclusion holds also in this case for the corresponding elements of reality and in conclusion, we may arrive to identify $E_{01}$, $E_{10}$, $E_{02}$, $E_{20}$.
However, note that we have

$$E_{01}E_{10} = E_{10}E_{01}\ ;\ E_{02}E_{20} = E_{20}E_{02}\ ;\ E_{01}\,E_{20} = E_{20}E_{01}\ ;\ E_{10}\,E_{02} = E_{02}\,E_{10}$$

but it is that (58)

$$E_{01}\,E_{02} \neq E_{02}\,E_{01} \quad \text{and} \quad E_{10}\,E_{20} \neq E_{20}\,E_{10}$$

**Thus we reach the conclusion that there are elements of reality which correspond to noncommuting quantum observables.**
On the other hand, quantum mechanics is unable to attribute simultaneously definite numerical values to these noncommuting quantum observables. Thus, in conclusion**, we have the EPR paradox or the EPR dilemma**, as Peres actually called it concluding the same kind of discussion in his paper [13].
In order to arrive to identify the actual root of the paradox and to solve it, let us go on and, still according to Peres [13], let us consider the product $E_{01}\,E_{20}$, connecting to it the product of the individual numerical values, $c_{01}c_{20}$, being $c_{01}$ and $c_{20}$ respectively the numerical values corresponding to $E_{01}$ and to $E_{20}$. In the same manner, let us define the numerical value of the product $E_{10}\,E_{02}$ as the product of the individual numerical values, $c_{10}c_{02}$, corresponding $c_{10}$ and $c_{02}$ respectively to $E_{10}$ and to $E_{02}$.
**From the previous discussion it results that these products must be equal as one may easily verify.**
On the order hand, we know also that they must be opposite since the singlet state in quantum physics gives that

$$(E_{21} + E_{12})\,\psi = 0 \quad \text{and} \quad E_{01}\,E_{20} = E_{21}\ ; \qquad E_{10}\,E_{02} = E_{12} \tag{59}$$

**So, we arrive to have here another explicit form of the paradox or also, as Peres called it at this stage [13], we have here an algebraic contradiction.**
According to Peres [13], note in particular that $E_{01}\,E_{20} + E_{10}\,E_{02}$ should represent still an element of reality, defined on the basis of the previous EPR's criterion, and thus, also following this line of reasoning, we find that still the paradox (or the dilemma, or still the algebraic contradiction ) hold.
Until here we have followed rigorously the argument that was exposed by Peres in 1992 [13] .
We may now see how the solution of the paradox is obtained.

## 3. The Solution of the Paradox

Let us explicit now the E.P.R. paradox using the Clifford algebra. We must consider $Cl_3$ members (basic elements) (with matrix representation at order n = 4 ), $E_{0i}$ (i = 1, 2, 3) to represent three-components of spin of the first particle and $E_{i0}$ (i = 1, 2, 3) to represent three-components of spin regarding the other particle. For details, see the (18), the (19), the (20), and the (24) of the previous section. We have thus two basic sets of elements ($E_{01}$, $E_{02}$, $E_{03}$) for the first particle, and ($E_{10}$, $E_{20}$, $E_{30}$) for the other particle.
Note now the important feature to use Clifford bare-bone skeleton of quantum mechanics.
Let us remember that, at the order n = 4, we have also other basic sets of elements, all those given by
($E_{01}$, $E_{12}$, $E_{13}$), ($E_{01}$, $E_{22}$, $E_{23}$), ($E_{01}$, $E_{32}$, $E_{33}$), ($E_{02}$, $E_{11}$, $E_{13}$), ($E_{02}$, $E_{21}$, $E_{23}$), ($E_{02}$, $E_{31}$, $E_{33}$), ($E_{03}$, $E_{11}$, $E_{12}$), ($E_{03}$, $E_{21}$, $E_{22}$), ($E_{03}$, $E_{31}$, $E_{32}$), ($E_{10}$, $E_{23}$, $E_{33}$), ($E_{10}$, $E_{22}$, $E_{32}$), ($E_{10}$, $E_{21}$, $E_{31}$), ($E_{20}$, $E_{13}$, $E_{33}$), ($E_{20}$, $E_{12}$, $E_{32}$), ($E_{20}$, $E_{11}$, $E_{31}$), ($E_{30}$, $E_{13}$, $E_{23}$), ($E_{30}$, $E_{12}$, $E_{22}$), ($E_{30}$, $E_{11}$, $E_{21}$). (60)
These are all the possible basic sets of elements that may be formed starting with $E_{0i}$ and $E_{i0}$, and, as one may verity, each basic set provides to the basic requirement that permutations of (1, 2, 3) must be involved in each set. Let us remember that this is a fundamental requirement of the basic elements since it follows directly from the noncommuting relations that must hold for the basic elements in each considered basic set.
Let us consider the profound modification respect to traditional quantum mechanics. Discussing E.P.R., we may consider also $E_{ij}$ basic elements that are essentially new, and that are not considered in the usual quantum mechanical formulation.
Now, let us return to the E.P.R. argument and let us impose to our basic sets the condition that our quantum system of two spin-½ particles is in a singlet state.
Let us consider that this physical condition is reduced in algebraic terms to be

$$E_{11} = E_{01}\,E_{10} = E_{10}\,E_{01}; \qquad E_{22} = E_{02}\,E_{20} = E_{20}\,E_{02}; \qquad E_{33} = E_{03}\,E_{30} = E_{30}\,E_{03} \tag{61}$$

with the attribution to $E_{11}$, to $E_{22}$, and to $E_{33}$ of the numerical value -1. According to the arguments developed in the previous section, we must have that

$$(\,E_{11} + 1\,)\,\psi = 0\ ; \qquad (E_{22} + 1)\psi = 0\ ; \quad (E_{33} + 1)\psi = 0 \tag{62}$$

where we obtain the idempotent $\psi$ to be $\psi_1\ \psi_2\ \psi_3$ with

$$\psi_1 = \left(\frac{E_{11}-1}{2}\right);\quad \psi_2 = \left(\frac{E_{22}-1}{2}\right);\quad \psi_3 = \left(\frac{E_{33}-1}{2}\right),\ \text{and}\ \psi_1\ \psi_2\ \psi_3 = \psi_2\ \psi_1\ \psi_3 = \psi_3\ \psi_1\ \psi_2 \qquad (63)$$

We verify that

$$E_{11}\ \psi = -\ \psi\ ; \qquad E_{22}\ \psi = -\ \psi\ ; \qquad E_{33}\ \psi = -\ \psi \qquad (64)$$

The (64) represent for us the basic algebraic formulation characterizing our E.P.R. system in the case of a quantum system composed of two spin -1/2 particles in a singlet state.

In order to proceed in the analysis of the E.P.R. paradox, we must now test the validity of the Clifford bare-bone quantum mechanical skeleton that we introduced in the previous section. We must remember that in the previous section, we introduced the (25) and the (26). We may say that we cannot expect doubts on the (25) and the (26) in the framework of Clifford algebra as well as with regard to the physical meaning that we attributed to such two expressions. Thus, we will not comment them in detail. Instead, soon after, we considered the (33) and, on the basis of the (33) and the (36), we obtained the (35) and the (38). The reader will remember here that we spoke of the rule of silence of usual quantum mechanics with regard to the importance that we attribute to the (35) and to the (38) in a bare-bone-skeleton of quantum theory while instead they are not usually considered in the traditional framework of quantum mechanics. Note that, by introducing the (35) and the (38), we suggested a new kind of quantum methodological formulation that, in our opinion, we should consider while it is absent in tradition formulation of quantum mechanics. It is now arrived the moment to test the validity of such new methodology applying it to the analysis of E.P.R. systems since they are the most investigated and the most known quantum mechanical systems. Analyzing E.P.R. systems by this methodology we, on one hand, will test if we will obtain conclusions in accord with our basic knowledge of such systems, and, on the other hand, we will consider the new conclusions that will emerge from this our elaboration, and then possibility to obtain a final explanation of the same nature of E.P.R. systems and thus the solution of the E.P.R. paradox. Thus, let us return to apply the methodology that we previously considered from the (25) to the (38).

Starting from the (62), we have to consider that

$$E_{0i}\ (E_{11} + 1)\ \psi = 0;\quad E_{i0}\ (E_{11} + 1)\ \psi = 0;\quad E_{ii}\ (E_{11} + 1)\psi = 0;\quad \text{and}\quad E_{ij}\ (E_{11} + 1)\ \psi = 0 \qquad (65)$$

and similar expressions for $E_{22}$ and, finally, for $E_{33}$. After calculations, we obtain that

$$E_{01}\ \psi = -\ E_{10}\ \psi = -\ i\ E_{23}\ \psi;\qquad E_{02}\ \psi = -\ E_{20}\ \psi = \ i\ E_{13}\ \psi;\ E_{03}\ \psi = \ i\ E_{21}\ \psi;\ E_{03}\ \psi = -\ E_{30}\ \psi;$$
$$E_{12}\ \psi = -\ E_{21}\ \psi\ ;\ E_{23}\ \psi = -\ E_{32}\ \psi\ ;\ E_{13}\ \psi = -\ E_{31}\ \psi \qquad (66)$$

The (66) represent all the algebraic relations that must hold when we consider a quantum system of two spin-½ particles in a singlet state and thus with connected the (62).The (66) represent the central point of our elaboration. They are essentially new. We outline here the importance to have expressed for the first time the (66) as direct consequence of the (62). In seventy years of research on E.P.R. correlated quantum systems, it was never evidenced previously that in correlated quantum E.P.R. systems obeying to the (62), the (66) must hold. Once again, thus, the (66) are indicative of the unevading importance to use Clifford algebra in formulation of quantum physical reality. We cannot avoid to consider and to respect the (66) when we are engaged in a discussion regarding quantum mechanics of correlated quantum systems with connected the (62).

Let us see in detail the (66). With exemplary clearness the (66) evidence not only the basic physical features of our considered quantum system but, in particular, the existing incompatibility between quantum physical reality and the criterion to fix elements of reality as instead it was posed by Einstein, Podolsky, and Rosen as basis of the paradox. Let us observe step by step the (66).

The conclusion that it must be $E_{0i} = -\ E_{i0}$ (i = 1, 2, 3) holds in perfect accord with the physical features of our considered EPR quantum system.

We have also that

$$E_{01} = -\ i\ E_{23}\ ;\qquad E_{02} = \ i\ E_{13}\ ;\qquad E_{03} = \ i\ E_{21} \qquad (67)$$

Soon after we have that always it must be that

$$E_{13} = -\ E_{31},\qquad E_{23} = -\ E_{32} \qquad (68)$$

and, in particular, it results that always it must be that

$$E_{12} = -\ E_{21} \qquad (69)$$

In order to avoid possible misinterpretations and errors, we repeat here that the (67), the (68), the (69), as well as all the previous and following algebraic expressions, must be considered, as in the (66), with regard to $\psi$.

Writing as example $E_{01} = -\ E_{10}$, we intend to say that this happens with regard to $\psi$.

Thus, in conclusion, examining the (64), that are imposed as the necessary condition of our system of two spin-½ particles to be in a singlet state, we find that necessarily, all of the (66) must hold, and, in particular, it must hold the (69). The reason by which the (69) is so important for us is that, with regard to $E_{12}$ and $E_{21}$, we concluded our argument of the previous section obtaining the paradox, or the **dilemma** or the **algebraic contradiction** [13], as final result. Let us remember that in the previous section we defined the numerical value of the product $E_{21}$ as the product of the individual numerical values of $E_{01}$ and $E_{20}$; we defined the numerical value of a product $E_{12}$ as the product of the individual numerical values of $E_{10}$ and $E_{02}$. From the discussion we found that these products must be equal while, on the basis of the (69), we find that they must be opposite. Here the paradox. The (69), now obtained only using our algebraic method, evidences, that, if the (62) holds (that is to say: if we have a system of two spin-½ particles in a singlet state), necessarily the (69) must hold and it results in excellent accord with the usual quantum mechanical

predictions, and with the results that were reported by Peres [13]. Thus, we have re-obtained the actual root of the paradox only using our algebraic method.
Let us comment our results only using our algebra.
Let us consider the two basic elements $E_{12}$ and $E_{21}$ since they regard directly the paradox, as they were introduced in this previous section. Mathematically speaking, it is evident that we may express $E_{12}$ as well as $E_{21}$ in some different manners. As example, we may consider that

$$E_{12} = E_{10}\, E_{02} \qquad \text{and} \qquad E_{21} = E_{20}\, E_{01} \tag{70}$$

In this case we are admitting existing "elements of reality" in accord to E.P.R. criterion and to the discussion of the previous section. However, let us see the impossibility of the (70). Since the (62) and the (66) must hold, we have that $E_{10} = - E_{01}$ and $E_{20} = - E_{02}$.. It becomes evident that the (70) cannot regard the physical reality of our considered quantum system of two spin-½ particles in a singlet state, since, according to the (70), we have that

$$E_{12} = E_{10}\, E_{02} = E_{01}\, E_{20} = E_{21}$$

and (71)

$$E_{21} = E_{20}\, E_{01} = E_{02}\, E_{10} = E_{12}$$

The (71) violate the (69). Thus, in order to respect the (69) of the (66), **we cannot escape to consider that $E_{12}$ and $E_{21}$ must necessarily contain an explicit expression of permutation of (1, 2, 3).** This is the central point from the mathematical view point. Transferring this net requirement from mathematical to physical terms, we find the profound physical implications and thus we have the final explanation of the paradox. Let us express always $E_{12}$ and $E_{21}$ by the basic elements $E_{10}\, E_{02}$ and $E_{20}\, E_{01}$ as in the (70) but this time expliciting permutations in the following manner:

$$E_{12} = \frac{E_{13}\, E_{01}}{i}$$

and (72)

$$E_{21} = \frac{E_{22}\, E_{03}}{i}$$

We see that in the first case the Clifford member may be written in the following manner

$$E_{12} = \frac{E_{10}\, E_{03}\, E_{01}}{i} = E_{10}\ \frac{E_{03}\, E_{01}}{i}; \qquad E_{02} = \frac{E_{03}\, E_{01}}{i} \tag{73}$$

while in the second case the Clifford member $E_{21}$ may be written in the following manner

$$E_{21} = \frac{E_{20}\, E_{02}\, E_{03}}{i} = E_{20}\ \frac{E_{02}\, E_{03}}{i}; \qquad E_{01} = \frac{E_{02}\, E_{03}}{i} \tag{74}$$

The substantial difference between the (70) and the (72), the (73) and the (74) is that in the (72) and the (74) a permutation of (1, 2, 3) is involved that instead is missing in the (70). In the (73) it is involved a permutation of (1, 2, 3) since we have $E_{03}\, E_{01}$ that represents $E_{02}$ while in the (74) it is involved a permutation of (1, 2, 3) since we have $E_{02}\, E_{03}$ to represent $E_{01}$. Since in our considered algebra, a permutation is connected with noncommuting basic elements while absence of permutation is connected with commuting basic elements, omitting to consider a permutation in the (70), that instead appears in the (73) and the (74), we perform the greatest error of physical analysis: we consider a commutation of basic elements (i.e., in correspondence, a commutation of quantum observables) where instead a noncommutation must be necessarily considered. As consequence, a paradox (or a dilemma or an algebraic contradiction) is induced because we attribute existing Einstein's "elements of reality", while instead they cannot be admitted. Let us remember in fact that, in our Clifford bare-bone skeleton of quantum mechanics as well as in standard quantum mechanics, commutation regards our ability to ascribe simultaneous unambigous numerical values to the considered basic elements, or, in correspondence, to the considered quantum mechanical operators, instead noncommutation regards our unability to ascribe simultaneous unambigous values to the same basic elements and, in correspondence, to operators in quantum mechanics.
It is immediate to verify, in fact, that, given two noncommuting basic elements, we cannot attribute simultaneous definite numerical values to such elements, and this is to say that we cannot find, as example, the same idempotent $\psi$ giving $E_{01}\, \psi = \pm\psi$ and $E_{02}\, \psi = \pm\psi$.
So in conclusion, a fixed number of non commuting relations is rigorously required in order to represent the basic quantum features of a given quantum system. This must be valid for all the quantum systems. In the case of a quantum system of two spin-½ particles in a singlet state, we arrive to write the (62) and the (66) that in some manner fix this strong physical-requirement of our system. As consequence the (62) and the (66) cannot be violated or overcome. If we consider commuting basic elements where instead the physical reality of the considered quantum system requires to consider noncommuting basic elements, we force our quantum system to assume physical features that actually it has not. And thus the paradox follows.
Let us now solve the (73) and the (74). We find that

$$E_{12} = \frac{E_{10}\, E_{03}\, E_{01}}{i} = \frac{E_{03}\ E_{10} E_{01}}{i} = \frac{-E_{03}}{i}; \tag{75}$$

$$E_{21} = \frac{E_{22}\, E_{03}}{i} = \frac{E_{20}\, E_{02}\, E_{03}}{i} = \frac{-E_{03}\, E_{02}\, E_{20}}{i} = \frac{E_{03}}{i}$$

as required by the (69). So the E.P.R. paradox of quantum mechanics vanishes.
It may be also seen that all the considered expressions, the (72), the (73), the (74), and the (75) are in perfect accord with each result contained in the (66). This indicates that we have now respected all the physical requirements that are connected to our considered quantum system, and so, rejecting in our case the (70) and the (71), that cannot be accepted, the E.P.R. paradox vanishes.
We may conclude outlining in particular the physical importance to have expressed $E_{02}$ and $E_{01}$ as in the (73) and the (74) respectively.
The expression

$$E_{02} = \frac{E_{03}\, E_{01}}{i}$$

in $E_{12} = E_{10}\, E_{02}$ of the (67) states that, if we consider $E_{10}$ in $E_{12}$, we cannot escape to consider the functional dependence of $E_{02}$ from $E_{03}$ and $E_{01}$. In the same manner, the presence of

$$E_{01} = \frac{E_{02}\, E_{03}}{i}$$

in $E_{21} = E_{20}\, E_{01}$ of the (68) states that, if we consider $E_{20}$ in $E_{21}$, we cannot escape to consider the functional dependence of $E_{01}$ from $E_{02}$ and $E_{03}$. These are mathematical and physical requirements of our considered quantum system obeying to the (62) and to the (66). As consequence, the unevading presence of such functional dependences forbids in an absolute manner to admit those independent elements of reality that instead the authors of E.P.R. paradox considered in their formulation.

## 4. Conclusions

In the previous sections we have exposed some elementary features of Clifford algebra realizing a preliminar and poor Clifford-bare-bone-skeleton of quantum theory that, however, contains some important results. We could define such results as a new stimulus that we may receive to reconsider the basic quantum mechanical foundations of standard theory.
We have obtained the expressions from the (25) to the (38) and soon after the (56) when considering the (53).
We know that, when we discuss quantum mechanics, we have so many interpretations: Copenhagen and standard interpretations, statistical interpretation, many-worlds, ……, and so on. It is the multitude of such recurring interpretations to indicate that some fundamental and deep problem still remains to the final acceptance of such theory. One reason could reside in the insufficiency of its mathematical and assiomatic formulation. Using only Clifford algebra, we have found the (35), the (38), and the (56) that standard quantum mechanics does not account while instead they are rigorously admitted through our algebraic formulation. In traditional quantum mechanics, as we said, a strange rule of silence instead holds during a measurement of an observable with regard to the simultaneous presence of other non commuting observables. Non commutativity of observables is the central pillar of the traditional quantum mechanics, and it still remains to be the central core in quantum bare-bone skeleton of Clifford algebra. However a problem remains in traditional formulation of the theory: how is it that we renounce to characterize non commutativity of observables, during a measurement, when instead we may conceive that it plays a central role also during a measurement? By Clifford algebra, expressed from the (25) to the (38) and the (56), we have seen that we may arrive to express and to characterize non commutativity for $e_1$, $e_2$, $e_3$ when, as example, measuring $e_3$. This is the first important conclusion that we submit. The second point is that we have shown the importance to use basic elements $E_{ij}$ that are substantially missing in traditional formulation of quantum mechanics.
Still the statement holds that non commutativity plays the central role in quantum dynamics of nature. In fact, we have discussed one of the basic paradoxes of quantum mechanics, the E.P.R. paradox. We have analyzed it on the basis of the poor Clifford mathematical rules that we have developed in the present paper, all rules based and deriving from non commutativity of basic elements of Clifford algebra. As conclusion, we have seen that, not only we have been able to give a Clifford mathematical exposition of E.P.R., but we have arrived also to indicate the intrinsic nature of such paradox, explaining the central role played from non commutativity: in fact, we have shown that, if we could neglect to consider non commutativity, we would obtain the accord with Einstein's predictions, while instead the quantum mechanical predictions are obtained only correctly considering non commutativity of the basic employed Clifford elements. Thus, if we have to give some credit to the Clifford-bare-bone-skeleton of quantum theory that we have formulated in this paper, we have just considering the important role of the (35), of the (38), and of the (56) from one hand to complete some basic foundations of standard quantum theory but, on the other hand, we have also to accept, with the only exception of such previously said insufficiences, the great predictive and correct power of this standard theory.
With regard of this problem we have to comment also on the actual value that we have to attribute to the Clifford-bare-bone-skeleton of quantum theory that we have formulated.
It must be clear that we have not given a bare-bone-skeleton of the theory; we have given only a poor bare-bone-skeleton of quantum theory. This difference is of substantial importance. Classical physics uses locality, continuity and determinism. Traditional quantum mechanics uses non locality, jumps, non commutation and indeterminism but it also reintroduces locality, continuity and determinism when it moves, as example, at the level of wave formulation. In the introduction of the present paper we outlined that conventionally formulated quantum mechanics start always using

classical analogies. Our bare-bone-skeleton of quantum theory is so poor since it has recovered here only some features of the more general quantum theory. Our elaboration has only recovered non locality, quantum jumps, non commutation and indeterminism. With regard to this last category, it was evidenced, in particular, the indeterminism to be an intrinsic feature of Clifford algebra by using the (48), while other important features were also considered by us elsewhere [5]. Thus, in conclusion, we may say **that our bare-bone-skeleton of quantum theory only recovers the basic quantum principles of reality**. This is a limit of our formulation but, with regard to the questions that we outlined in the introduction of the present paper, this limit becomes soon after an important feature of our elaboration. Let us analyze the manner in which we have developed our formulation.

In brief, **we have introduced only mathematical arguments, and we have delineated as such mathematical elaboration reflects the fundamental quantum principles.**

**Actually, we have used only the basic axioms of Clifford algebra**.

Thus, a conclusion seems evident: **in the physical reality the quantum principles are a manifestation of the basic axioms of Clifford algebra**.

This conclusion, according to the arguments that we developed in the introduction, seems to be of relevant interest. Let us make deeper our analysis.

It is well known that Clifford algebra also exhibits some other important physical features. When, as example, the (3) is written in the following form

$$q = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

we may say that to the Clifford member $q \in Cl_3$ there corresponds a point $P(x_\mu)$ ($\mu$ = 0, 1, 2. 3) in a real four-dimensional space that results to be identical to the Minkowski-space as just it was outlined by K Imaeda [14] in 1976. In this manner we arrive to the conclusion that, on one hand, quantum principles are a manifestation of the basic set of axioms of Clifford algebra, and, on the other hand, the space-time of physical events that always we retained to represent the primary structure that realizes physical events, must not be retained to be primary. In fact, basic set of axioms of Clifford algebra gives Minkowski space-time as its derivation. We have a primary algebraic axiomatic structure that manifests quantum principles, and precedes space-time. Let us introduce some positions that support this our conclusion.

Let us consider Eddington's words in 1937 [15]. "It is a fallacy to think that the conception of location in space-time based on observations of large scale phenomena can be applied unmodified to happenings which involve a small number of quanta. The phenomena are excluded from the outset by the adoption of a co-ordinate frame of reference".

Schwinger [16] wrote in 1958: "A convergent theory cannot be formulated consistently within a framework of the present space-time concepts."

Penrose writes in 1979 [17]. "Space-time theory would be expected to arise out of some more primitive combinatorial theory".

Einstein wrote in 1936 [18]: "the success of the Heisenberg method points to a purely algebraic method of the description of nature, that is, to the elimination of continous functions from physics. Then, however, we must give up, in principle, the space-time continuum."

Bohr in 1925 wrote [19]: "…. I believe that these difficulties so thoroughly rule out the retention of the ordinary space-time description of phenomena."

B.J. Hiley [20] repeatedly discussed the important role of Clifford algebra for quantum processes and repeatedly outlined that space-time must not be considered primary. His words illuminate he wrote in 2000: "If space-time is taken as primary, then, ipso facto, locality is absolute. Indeed the space-time manifold dominates classical physics because it has locality built into it right at the beginning. If we retain the space-time manifold, then quantum non locality sits very uncomfortable in such a structure". And he continues:

"Could it be that our insistence on taking a given space-time as basic is at fault? Could space-time merely be an appearance, a feature that has to be abstracted from some deeper structure, a structure where space-time itself is not taken as basic? If this were the case, then it would be establishing locality that would present the problem. Could it be that locality itself is merely relationship? This relationship dominates the macroscopic world, but it would not be universally valid at the quantum level. Yes there is relativity, but does that theory apply to the level of a single photon or only to a statistical ensemble of photons?".

Let us admit that locality is a relationship, according still to Hiley, and as shown in this paper, quantum phenomena should arise owing to the existing basic Clifford axioms and thus they are, in some sense, beyond our space-time that, as we have seen, still derives from the basic set of Clifford axioms but it must be considered as a mere projection starting with the actual space of Clifford algebra. "Quantum phenomena are projected into space-time by our macroscopic instruments", [20] Hiley in 1991 spoke of pre-space and Wheeler [21] in 1980 spoke of pre-geometry to characterize that quantum processes should evolve in some more general space. We think to have identified such pre-space or such pre-geometry in the basis axiomatic Clifford algebra that contains quantum principles and it generates traditional space-time as its projection. In this manner, the space-time of the classical world, still according to Hiley, [20], "becomes a statistical approximation and not all the quantum processes can be projected into this space without producing paradoxes" pertaining, as example, "non separability and non locality". In classical physics instead, since everything is local, the single space-time provides descriptions that result to be free from contradictions, see ref. [20].

In this manner the Clifford bare bone skeleton of quantum theory, no more results to be a poor skeleton of quantum mechanics but the basic axiomatic framework in which quantum phenomena should be actually realized.

In this manner, the basic scheme of Wheeler [21] should be confirmed that **the first day** was quantum principles, that we attribute to basic axiomatic Clifford algebra, and **the second day** was classical geometry that we attribute to basic space-time projection of such axiomatic Clifford structure.

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